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Painlevé test of coupled Gross–Pitaevskii equations

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Abstract

A Painlevé (P) test of the coupled Gross–Pitaevskii equations has been carried out with the result that the coupled equations pass the P test only if a special relation containing system parameters (masses and scattering lengths) is satisfied. Computer algebra is applied to evaluate the $j = 4$ compatibility condition for admissible external potentials. The appearance of an arbitrary real potential embedded in the external potentials is shown to be the consequence of the coupling. The connection with recent experiments related to the stability of two-component Bose–Einstein condensates of Rb atoms is discussed.

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1. Introduction

Recently there has been a growing interest in the Gross–Pitaevskii (GP) equations [1, 2] describing two-component Bose–Einstein condensates (BECs) in external trap potentials [3–22]. In the absence of the confining potential, the GP equations reduce to the coupled nonlinear Schrödinger (NLS) equations, which play an important role in optics [23]. Coupled GP equations are also used to describe Josephson-type oscillations between two coupled BECs [9–11] or the spin-mixing dynamics of spinor BECs [12–15], or to explore such interesting field of matter waves as possible atomic soliton lasers [4, 24, 25].

An efficient tool of the analysis of the nonlinear partial differential equations is the Painlevé (P) method [26,27], which serves to explore the singularity structure of the underlying equations, and establish integrability conditions [28]. The P analysis of the single NLS equation has been performed by Steeb *et al* [29], and the damped NLS (or the GP) equation has been investigated by Clarkson [30]. A fairly large class of coupled NLS equations including third-order dispersions has been analysed by Radhakrishnan *et al* [31]. Recently the symmetrically coupled higher-order NLS equations have been tested by using the P method [32].

Because of the experimental developments in forming two-component BECs [16] and the possibility of confining BECs in a linear shape [33], we shall perform the P test of the coupled one-dimensional GP equations in order to establish certain necessary conditions of integrability. (The term integrability is used here in the general sense [27, 28] involving the

P property and soliton formation.) The results obtained for the trap potentials are similar to those found by Clarkson [30] in the case of the damped NLS equations: the trap potential should be linear and/or quadratic in the coordinate variable x. In the quadratic case a source term depending only on time t should also be present in the external potential $V(x, t)$.

A novel feature of our analysis is the possibility of the appearance of an arbitrary common potential term $\tilde{V}(x, t)$ within the confining potentials $V_1(x, t)$ and $V_2(x, t)$. Its presence may prove useful for fine tuning experiments with two-component BECs. We consider the system of coupled GP equations in its most general form containing different masses, external potentials and mutual coupling strengths. As a result we shall derive compatibility conditions, the fulfillment of which depends on the parameters characterizing the GP equations. We show that in a particular experiment [7], employing two hyperfine states of Rb atoms as components of the BEC, the vortex stability corresponds to just the parameter ratios satisfying our general formula derived in this paper.

The organization of this paper is as follows. In section 2 the P analysis of two coupled GP equations will be carried out including the determination of the leading orders, the recursion relations, the resonances and the compatibility conditions. The consequences of the compatibility relations for the potentials are discussed in section 3, where other consistency requirements are also studied. In section 4 we make comparisons with earlier results and investigate compatibilities with existing experimental and numerical findings related to two-component BECs. Section 5 is devoted to a short summary.

2. Painlevé test

Let us consider the following (1 + 1)-dimensional inhomogeneous NLS equations for the wavefunctions ψ_1 and ψ_2 with the external potentials $U_1(x, t)$ and $U_2(x, t)$:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_1(x,t) = \left(-\frac{\hbar^2}{2m_1}\nabla^2 + U_1(x,t) + U_{11}\,|\psi_1(x,t)|^2 + U_{12}\,|\psi_2(x,t)|^2\right)\psi_1(x,t) + U_{10} \tag{2.1a}$$

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi_2(x,t) = \left(-\frac{\hbar^2}{2m_2}\nabla^2 + U_2(x,t) + U_{21}\,|\psi_1(x,t)|^2 + U_{22}\,|\psi_2(x,t)|^2\right)\psi_2(x,t) + U_{20} \tag{2.1b}$$

which, in the absence of the inhomogeneities U_{10} and U_{20}, are commonly called the coupled GP equations [1, 2].

Here m_i denotes the mass of the atomic species i ($i = 1, 2$) of the two-component BEC gas and U_{ij} is related to the interactions between the atoms i and j ($i, j = 1, 2$) via the relation $U_{ij} = 2\pi\hbar^2 a_{ij} N_j / A\,\mu_{ij}$, where N_j means the number of atoms in the jth component of the BEC, a_{ij} is the scattering length characterizing the interaction between atoms i and j, A represents a general cross sectional area confining species i and j and $\mu_{ij} = m_i m_j/(m_i + m_j)$ is the reduced mass.

By introducing the new parameters

$$\lambda = \frac{\hbar}{2m_1} \qquad \vartheta = \frac{\hbar}{2m_2} \qquad T_{ij} = \frac{1}{\hbar}U_{ij} \qquad (i, j = 1, 2) \tag{2.2a}$$

and notations

$$u = \psi_1 \qquad w = \psi_2 \qquad V_i = \frac{1}{\hbar}U_i \qquad V_{i0} = \frac{1}{\hbar}U_{i0} \qquad (i = 1, 2) \tag{2.2b}$$

we write the GP equations into the standard form of the P analysis

$$\mathrm{i}u_t + \lambda u_{xx} - T_{11}\,|u|^2\,u - T_{12}\,|w|^2\,u = V_1 u + V_{10} \tag{2.3a}$$

$$\mathrm{i}w_t + \vartheta w_{xx} - T_{21}|u|^2 w - T_{22}|w|^2 w = V_2 w + V_{20} \tag{2.3b}$$

where T_{ij} and λ, ϑ represent, as defined by equation (2.2a), the interaction and mass parameters, respectively.

In order to apply the P analysis, we first complexify all variables to obtain equations (2.3a) and (2.3b) in the form ($v = u^*$, $z = w^*$)

$$\mathrm{i}u_t + \lambda u_{xx} - T_{11}u^2 v - T_{12}wzu = V_1 u + V_{10} \tag{2.4a}$$

$$-\mathrm{i}v_t + \lambda v_{xx} - T_{11}uv^2 - T_{12}wzv = V_1^* v + V_{10}^* \tag{2.4b}$$

$$\mathrm{i}w_t + \vartheta w_{xx} - T_{21}uvw - T_{22}w^2 z = V_2 w + V_{20} \tag{2.4c}$$

$$-\mathrm{i}z_t + \vartheta z_{xx} - T_{21}uvz - T_{22}wz^2 = V_2^* z + V_{20}^* \tag{2.4d}$$

where the functions u, v, w and z are treated as independent complex functions of the complex variables x and t, and $V_1^*(x,t)$, $V_{10}^*(x,t)$, $V_2^*(x,t)$ and $V_{20}^*(x,t)$ are formal complex conjugates of $V_1(x,t)$, $V_{10}(x,t)$, $V_2(x,t)$ and $V_{20}(x,t)$, respectively.

The next step is to seek the solutions of (2.4a)–(2.4d) in the form

$$u(x,t) = \phi^p(x,t)\sum_{j=0}^{\infty} u_j(t)\phi^j(x,t) \qquad v(x,t) = \phi^q(x,t)\sum_{j=0}^{\infty} v_j(t)\phi^j(x,t) \tag{2.5a}$$

$$w(x,t) = \phi^r(x,t)\sum_{j=0}^{\infty} w_j(t)\phi^j(x,t) \qquad z(x,t) = \phi^s(x,t)\sum_{j=0}^{\infty} z_j(t)\phi^j(x,t) \tag{2.5b}$$

with the Kruskal ansatz

$$\phi(x,t) = x - \xi(t) \tag{2.6}$$

and $\xi(t)$, $u_j(t)$, $v_j(t)$, $w_j(t)$ and $z_j(t)$, $j = 0, 1, 2, \ldots$, being analytic functions of t in the neighbourhood of a noncharacteristic movable singularity manifold defined by $\phi = 0$. Similarly, the external potential V_i confining specimen i is also expanded about the singularity manifold $\phi = 0$ as follows ($i = 1, 2$):

$$V_i(x,t) = \sum_{j=0}^{\infty} V_{i,j}(t)\phi^j(x,t) \qquad V_{i,j}(t) = \frac{1}{j!}\left(\frac{\partial^j V_i(x,t)}{\partial x^j}\right)_{x=\xi(t)}. \tag{2.7}$$

Substituting expansions (2.5a), (2.5b) and (2.7) into equations (2.4a)–(2.4d) and equating like powers of ϕ we obtain:

(i) equations for determining the leading orders p, q, r and s and
(ii) recursion relations for deriving the functions u_j, v_j, w_j and z_j.

In order that equations (2.4a)–(2.4d) pass the Painlevé test it is required that the numbers p, q, r and s be non-positive integers. Moreover, the recursion relations should be consistent in all orders of j including the resonances.

2.1. Determination of the leading orders

To determine the leading orders p, q, r and s appearing in the expansions (2.5a) and (2.5b), it is sufficient to consider the expansion up to the zeroth order, $j = 0$. By substituting this truncated version of expansions (2.5a) and (2.5b) into (2.4a) we obtain

$$\begin{aligned}&\mathrm{i}u_{0,t}\phi^p + \mathrm{i}u_0 p\phi^{p-1}\phi_t + \lambda u_0 p(p-1)\phi^{p-2} - T_{11}u_0^2 v_0\phi^{2p+q} - T_{12}u_0 w_0 z_0\phi^{p+r+s}\\ &\quad = V_1 u_0\phi^p + V_{10}.\end{aligned} \tag{2.8a}$$

Three completely similar expressions arise from the substitution of the truncated version of expansions (2.5*a*) and (2.5*b*) into the remaining three equations (2.4*b*)–(2.4*d*):

$$-\mathrm{i}v_{0,t}\phi^q - \mathrm{i}v_0 q\phi^{q-1}\phi_t + \lambda v_0 q(q-1)\phi^{q-2} - T_{11}v_0^2u_0\phi^{2q+p} - T_{12}v_0w_0z_0\phi^{q+r+s} = V_1^* v_0\phi^q + V_{10}^* \tag{2.8b}$$

$$\mathrm{i}w_{0,t}\phi^r + \mathrm{i}w_0 r\phi^{r-1}\phi_t + \vartheta w_0 r(r-1)\phi^{r-2} - T_{21}u_0v_0w_0\phi^{p+q+r} - T_{22}w_0^2z_0\phi^{2r+s} = V_2 w_0\phi^r + V_{20} \tag{2.8c}$$

$$-\mathrm{i}z_{0,t}\phi^s - \mathrm{i}z_0 s\phi^{s-1}\phi_t + \vartheta z_0 s(s-1)\phi^{s-2} - T_{21}u_0v_0z_0\phi^{p+q+s} - T_{22}w_0z_0^2\phi^{r+2s} = V_2^* z_0\phi^s + V_{20}^*. \tag{2.8d}$$

By requiring the leading-order terms of equations (2.8*da*)–(2.8*dd*) to vanish one obtains the following equations:

$$\lambda p(p-1) = T_{11}u_0v_0 + T_{12}w_0z_0 \tag{2.9a}$$

$$\lambda q(q-1) = T_{11}u_0v_0 + T_{12}w_0z_0 \tag{2.9b}$$

$$\vartheta r(r-1) = T_{21}u_0v_0 + T_{22}w_0z_0 \tag{2.9c}$$

$$\vartheta s(s-1) = T_{21}u_0v_0 + T_{22}w_0z_0 \tag{2.9d}$$

and

$$p + q = -2 \tag{2.10a}$$

$$r + s = -2 \tag{2.10b}$$

from which the leading orders can uniquely be determined to be

$$p = q = r = s = -1. \tag{2.11}$$

For later use we infer from equations (2.9*a*)–(2.9*d*) the useful relation

$$\begin{pmatrix} u_0v_0 \\ w_0z_0 \end{pmatrix} = \frac{2}{\Delta}\begin{pmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{pmatrix}\begin{pmatrix} \lambda \\ \vartheta \end{pmatrix} \tag{2.12}$$

with $\Delta = T_{11}T_{22} - T_{12}T_{21}$. If accidentally $\Delta = 0$ happens then we may use the relation $u_0v_0/w_0z_0 = \mathrm{const}$ instead of (2.12), which case needs a special consideration.

2.2. Recursion relations

The next step of the P analysis is to again substitute expansions (2.5*a*), (2.5*b*) and (2.7) with the leading orders $p = q = r = s = -1$ into equations (2.4*a*)–(2.4*d*). After some algebra we obtain the recursion relations

$$\underbrace{\begin{pmatrix} Q_1 & -T_{11}u_0^2 & -T_{12}u_0z_0 & -T_{12}u_0w_0 \\ -T_{11}v_0^2 & Q_1 & -T_{12}v_0z_0 & -T_{12}v_0w_0 \\ -T_{21}v_0w_0 & -T_{21}u_0w_0 & Q_2 & -T_{22}w_0^2 \\ -T_{21}v_0z_0 & -T_{21}u_0z_0 & -T_{22}z_0^2 & Q_2 \end{pmatrix}}_{Q(j)}\begin{pmatrix} u_j \\ v_j \\ w_j \\ z_j \end{pmatrix} = \begin{pmatrix} F_j \\ G_j \\ H_j \\ K_j \end{pmatrix} \tag{2.13a}$$

where $j = 1, 2, \ldots$ and

$$Q_1 = \lambda(j-1)(j-2) - 2T_{11}u_0v_0 - T_{12}w_0z_0 \tag{2.13b}$$

$$Q_2 = \vartheta(j-1)(j-2) - 2T_{22}w_0z_0 - T_{21}u_0v_0 \tag{2.13c}$$

$$F_j = -\mathrm{i}u_{j-2,t} - \mathrm{i}(j-2)u_{j-1}\phi_t + \sum_{m=1}^{j-1}(T_{11}u_mu_{j-m}v_0 + T_{12}u_mw_0z_{j-m})$$

$$+\sum_{l=1}^{j-1}\sum_{m=0}^{l}(T_{11}u_m u_{l-m}v_{j-l}+T_{12}u_m w_{j-l}z_{l-m})$$
$$+\sum_{l=0}^{j-2}V_{1,l}u_{j-l-2}+V_{10,j-3}. \qquad (2.13d)$$

Here we use the notation that whenever an index is less than zero the expression itself is zero (for example, $V_{10,j-3}\equiv 0$ for $j\leqslant 2$). Furthermore G_j is obtained from F_j by interchanging u_l and v_l and letting $\mathrm{i}\rightarrow -\mathrm{i}$, $V_{1,l}\rightarrow V_{1,l}^*$ and $V_{10,l}\rightarrow V_{10,l}^*$. The expressions H_j and K_j can be obtained, respectively, from F_j and G_j by interchanging u_l and w_l, v_l and z_l, T_{12} and T_{21}, T_{11} and T_{22}, and letting $V_1\rightarrow V_2$ and $V_{10}\rightarrow V_{20}$.

The expressions F_j, G_j, H_j and K_j at a given j depend only on the expansion coefficients u_l, v_l, w_l and z_l with $l<j$. Therefore the equation (2.13*a*) represents recursion relations for the determination of the unknowns u_j, v_j, w_j and z_j from the knowledge of the prior calculated coefficient functions u_l, v_l, w_l and z_l with $l<j$.

2.3. Resonances

The above recursion relations (2.13*a*) determine the unknown expansion coefficients uniquely unless the determinant of the matrix $Q(j)$ is zero. Those values of j at which the determinant $\det(Q(j))$ becomes zero are called resonances. After some calculation one obtains

$$\det(Q(j))=\lambda^2\vartheta^2(j+1)\,j^2(j-3)^2(j-4)\left(j^2-3j+4-2\frac{\vartheta T_{11}u_0v_0+\lambda T_{22}w_0z_0}{\lambda\vartheta}\right) \qquad (2.14)$$

so the resonances of the coupled GP equations (2.3*a*) and (2.3*b*) are as follows:

$$j_{\mathrm{res}}=\{-1,0,0,3,3,4,j_1,j_2\}. \qquad (2.15)$$

Here j_1 and j_2 are the roots of the expression contained in the last parentheses of equation (2.14) and can formally be given as

$$j_{1,2}=\frac{3}{2}\pm\frac{1}{2}\sqrt{8\,\frac{\vartheta T_{11}u_0v_0+\lambda T_{22}w_0z_0}{\lambda\vartheta}-7}\quad\in\mathbb{Z}. \qquad (2.16)$$

As indicated, the resonances j_1 and j_2 must be integers so that the square root must be odd integers. From this one obtains a condition

$$\sqrt{8\,\frac{\vartheta T_{11}u_0v_0+\lambda T_{22}w_0z_0}{\lambda\vartheta}-7}=2m+1 \qquad m=0,1,2,\ldots \qquad (2.17)$$

involving the characteristic parameters T_{ij}, λ and ϑ of the GP equations (2.3*a*) and (2.3*b*). The number m can be considered as a classification number which classifies possible external potential families for which the system (2.3*a*), (2.3*b*) is integrable (in the general sense of integrability [27, 28]).

By re-arranging (2.17) and using relation (2.12) one obtains a more explicit condition necessary for any coupled GP equations to pass the P test:

$$\frac{2\,T_{11}T_{22}-(\vartheta/\lambda)\;T_{11}T_{12}-(\lambda/\vartheta)\;T_{21}T_{22}}{T_{11}T_{22}-T_{12}T_{21}}=\frac{1}{16}\left[(2m+1)^2+7\right] \qquad m=0,1,2,\ldots. \qquad (2.18)$$

It is also clearly seen that this expression depends only on the ratios λ/ϑ, T_{11}/T_{21} and T_{12}/T_{22} involving the characteristic parameters of the GP equations.

In summary, any coupled system of GP equations (2.3a), (2.3b) passes the Painlevé test only if its characteristic parameters λ, ϑ and T_{ij} $(i, j = 1, 2)$ obey the relation (2.18), otherwise it is probably not integrable. (See the discussions about the connection of the P test with integrability in [27, 28].)

2.4. *Compatibility conditions*

At each element of j_{res}, the recursion relations (2.13a) cannot be used for the calculation of the expansion coefficients. At these indices arbitrary functions may arise in the expansions (2.5a) and (2.5b). However, in order that the solution be expressible in the form of the expansions (2.5a), (2.5b) and (2.7), the recursion relations should be identically satisfied at $j \in j_{\mathrm{res}}$. The investigation of these specific requirements leads to relations called compatibility conditions which impose restrictions for the external potentials $V_i(x, t)$, $i = 1, 2$. We note that only the positive resonances are of interest.

2.4.1. Compatibility condition belonging to resonance $j = 3$. Let us consider equations (2.13a) at $j = 3$ and use equation (2.12). The result is an equation

$$\underbrace{\begin{pmatrix} -T_{11}u_0v_0 & -T_{11}u_0^2 & -T_{12}u_0z_0 & -T_{12}u_0w_0 \\ -T_{11}v_0^2 & -T_{11}u_0v_0 & -T_{12}v_0z_0 & -T_{12}v_0w_0 \\ -T_{21}v_0w_0 & -T_{21}u_0w_0 & -T_{22}w_0z_0 & -T_{22}w_0^2 \\ -T_{21}v_0z_0 & -T_{21}u_0z_0 & -T_{22}z_0^2 & -T_{22}w_0z_0 \end{pmatrix}}_{Q(3)} \begin{pmatrix} u_3 \\ v_3 \\ w_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} F_3 \\ G_3 \\ H_3 \\ K_3 \end{pmatrix} \tag{2.19}$$

whose matrix $Q(3)$ has rank two. Indeed, by multiplying the first row with v_0, the second with u_0, one obtains a matrix which possesses identical elements in its first two rows. Performing similar manipulations, one can make the third and fourth rows also identical. It then follows that the above recursion relation can only be consistent if the following compatibility conditions hold:

$$F_3v_0 = G_3u_0 \tag{2.20a}$$

$$H_3z_0 = K_3w_0. \tag{2.20b}$$

We emphasize that the above conditions are not independent from each other because, for example as shown by (2.13d), F_3 contains elements w_i and z_i with $i \leqslant 3$. Similarly, G_3, H_3 and K_3 also contain all types of expansion coefficient u_i, v_i, w_i and z_i with $i \leqslant 3$.

2.4.2. Compatibility condition belonging to resonance $j = 4$. By taking the recursion relations (2.13a) at $j = 4$ and applying equation (2.12), one arrives at the following equation:

$$\underbrace{\begin{pmatrix} 4\lambda - T_{11}u_0v_0 & -T_{11}u_0^2 & -T_{12}u_0z_0 & -T_{12}u_0w_0 \\ -T_{11}v_0^2 & 4\lambda - T_{11}u_0v_0 & -T_{12}v_0z_0 & -T_{12}v_0w_0 \\ -T_{21}v_0w_0 & -T_{21}u_0w_0 & 4\vartheta - T_{22}w_0z_0 & -T_{22}w_0^2 \\ -T_{21}v_0z_0 & -T_{21}u_0z_0 & -T_{22}z_0^2 & 4\vartheta - T_{22}w_0z_0 \end{pmatrix}}_{Q(4)} \begin{pmatrix} u_4 \\ v_4 \\ w_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} F_4 \\ G_4 \\ H_4 \\ K_4 \end{pmatrix}. \tag{2.21}$$

In the general case the matrix $Q(4)$ has rank three, which means that only three of its rows are independent. Using this fact, after some calculation we obtain the following compatibility condition:

$$T_{21}(F_4v_0 + G_4u_0) + T_{12}(H_4z_0 + K_4w_0) = 0. \tag{2.22}$$

We should investigate also the possibility when rank $(Q(4)) = 2$. In this case the compatibility condition decomposes into two distinct parts as can be seen in the following way. The rank of a matrix is equal to the maximal order of its nonsingular submatrices. We should thus calculate the determinants of all third-order submatrices of $Q(4)$ and investigate the cases when they simultaneously become zero. After a simple but lengthy calculation the following results are obtained for the determinants of the four third-order submatrices of the matrix $Q(4)$:

$$\left\{16\lambda\vartheta T_{21}u_0z_0;\ -16\lambda\vartheta T_{12}u_0w_0;\ 16\lambda\vartheta T_{12}w_0z_0;\ -16\lambda\vartheta T_{21}u_0^2\right\}. \tag{2.23}$$

Because λ and ϑ are the nonzero mass parameters (see definitions (2.2*a*)), it is clear that the subdeterminants vanish only if $T_{12} = T_{21} = 0$. This situation, however, corresponds to two uncoupled GP equations. We have thus arrived at the compatibility condition found by Clarkson [30] for the single GP equations:

$$F_4v_0 + G_4u_0 = 0 \tag{2.24a}$$

$$H_4z_0 + K_4w_0 = 0. \tag{2.24b}$$

We note, however, that the general compatibility condition, as given by equation (2.22), leads to external potentials (discussed in the next section) more complicated than that obtainable from equations (2.24*ba*), (2.24*bb*) with $T_{12} = T_{21} = 0$.

2.4.3. Compatibility condition belonging to resonances j_1 and j_2. We now consider the matrix $Q(j_{1,2})$ with $j_{1,2}$ taken from equation (2.16). In general the matrix $Q(j_{1,2})$ has rank three, from which the following compatibility condition arises:

$$\vartheta w_0z_0(F_{j_{1,2}}v_0 + G_{j_{1,2}}u_0) - \lambda u_0v_0(H_{j_{1,2}}z_0 + K_{j_{1,2}}w_0) = 0. \tag{2.25}$$

As before we analyse also the case when $\mathrm{rank}\big(Q(j_{1,2})\big) = 2$. After a lengthy but simple calculation we obtain the determinants of the four third-order submatrices of the matrix $Q(j_{12})$ to be

$$\begin{aligned}\big\{&T_{21}\Delta^5\,(u_0v_0)^3\,(w_0z_0)^2\,z_0;\ -T_{21}\Delta^5\,(u_0v_0)^3\,(w_0z_0)^2\,;\\ &T_{21}\Delta^5\,(u_0v_0)^3\,(w_0z_0)^2\,z_0;\ -T_{12}\Delta^6\,(u_0v_0w_0z_0)^3\,\big\}.\end{aligned} \tag{2.26}$$

Now, as clearly seen the determinants (2.26) vanish simultaneously only if $T_{12} = T_{21} = 0$ (or $\Delta = 0$, which case is not considered here). On the other hand, for this decoupled case one can determine the values j_1 and j_2 by using definition (2.16) and relation (2.12) to be $j_1 = 4$ and $j_2 = -1$. However, then, as can be checked easily by using (2.13*a*), the corresponding compatibility conditions reduce to those already discussed in connection with equations (2.24*ba*) and (2.24*bb*). We note however that, depending on the experimental situations, it is possible to obtain resonance values j_1 and j_2 greater than four. We should then use equation (2.25) for drawing conclusions about the admissible form of the external potentials.

3. Possible forms of the external potentials

In the preceding section we found equations, called compatibility conditions, that must be fulfilled in order for the GP equations to pass the P test. In this section we exploit the consequences of these equations for the general form of the external potentials V_1, V_{10}, V_2 and V_{20} appearing in equations (2.1*a*), (2.1*b*) and (2.3*a*), (2.3*b*). The experimental realization of such potentials may lead to detection of stable structures (such as vortices) in BECs.

Although the compatibility conditions are related to indices j at which the recursion relations (2.13*a*) do not apply to the calculation of the unknown coefficients $u_j(t)$, $v_j(t)$, $w_j(t)$ and $z_j(t)$, the equations (2.20*a*), (2.20*b*), (2.22) and (2.25) can be reduced to expressions in which only the zeroth-order coefficient functions $u_0(t)$, $v_0(t)$, $w_0(t)$ and $z_0(t)$ are present. This is because, at $j \notin j_{\mathrm{res}}$, use of the recursion relations (2.13*a*) and the definition (2.13*d*) of the functions F_j, G_j, H_j and K_j leads always to expansion coefficients u_j, v_j, w_j and z_j that are expressed by the zeroth-order functions $u_0(t)$, $v_0(t)$, $w_0(t)$ and $z_0(t)$.

In the following we shall present the results of the calculation belonging to each compatibility condition. For the resonance $j = 3$ the calculation can be performed easily by hand, but for $j = 4$ the computer program Maple [34] had to be invoked in order to perform the analytic manipulations. As a result of the Maple program, all the coefficients which multiply the higher-order powers of ϕ_t proved to be analytically zero. The expression associated with the zeroth-order power of ϕ_t has been evaluated further by hand to obtain the final results, which will be presented and discussed below.

3.1. Conditions for the potentials arising from $j = 3$

In section 2 it has been shown that at $j = 3$ the compatibility condition decomposes into two distinct parts, which are however not independent of each other (see equations (2.20*a*) and (2.20*b*) and the remark thereafter).

The elaboration of the compatibility conditions (2.20*a*) and (2.20*b*) related to $j = 3$ yields the relations

$$F_3 v_0 - G_3 u_0 = 0 \longrightarrow (V_{1,1} - V_{1,1}^*) u_0 v_0 + V_{10,0} v_0 - V_{10,0}^* u_0 = 0 \tag{3.1a}$$

$$H_3 z_0 - K_3 w_0 = 0 \longrightarrow (V_{2,1} - V_{2,1}^*) w_0 z_0 + V_{20,0} z_0 - V_{20,0}^* w_0 = 0. \tag{3.1b}$$

It is clear from equation (2.12) that only the products $u_0 v_0$ and $w_0 z_0$ are determined uniquely so that one element of each pair can be chosen arbitrarily. With the choices $u_0 = 1$, $w_0 = 1$ the above relations can only be satisfied if

$$V_{1,1} - V_{1,1}^* = 0 \qquad \text{and} \qquad V_{10,0} = V_{10,0}^* \equiv 0 \tag{3.2a}$$

$$V_{2,1} - V_{2,1}^* = 0 \qquad \text{and} \qquad V_{20,0} = V_{20,0}^* \equiv 0. \tag{3.2b}$$

These conditions show that the expansion coefficients $V_{1,1}$ and $V_{2,1}$ are real. Moreover, using the definition (2.7) for the expansion coefficients $V_{i,j}$ and the results (3.2), we obtain

$$0 = V_{10,0} = \frac{1}{0!} \left. \frac{\partial^0 V_{10}(x,t)}{\partial x^0} \right|_{x=\xi(t)} = V_{10}(\xi(t), t). \tag{3.3}$$

Since this equality holds for any arbitrary function $\xi(t)$, it follows that $V_{10}(x, t)$ must vanish. Similar argumentation leads to disappearance of $V_{20}(x, t)$.

In summary we conclude that in order for the equations (2.3*a*) and (2.3*b*) to pass the P test the inhomogeneity terms must vanish and the first-order expansion coefficient of the external potentials must be real:

$$V_{10} = V_{20} = 0 \tag{3.4a}$$

$$V_{1,1} = V_{1,1}^* \qquad \text{and} \qquad V_{2,1} = V_{2,1}^*. \tag{3.4b}$$

3.2. Conditions for the potentials arising from $j = 4$

Without presenting the details of the algebraic manipulations, we state that the compatibility condition (2.22) (partly with the aid of the formula manipulation program Maple) leads to the relation

$$\frac{T_{21}}{2\lambda}u_0v_0(V_{1,0}-V^*_{1,0})^2+\frac{T_{12}}{2\vartheta}w_0z_0(V_{2,0}-V^*_{2,0})^2$$
$$-T_{12}w_0z_0(V_{2,2}+V^*_{2,2})-T_{21}u_0v_0(V_{1,2}+V^*_{1,2})$$
$$-\mathrm{i}\frac{T_{21}}{2\lambda}u_0v_0\frac{\partial}{\partial t}(V_{1,0}-V^*_{1,0})-\mathrm{i}\frac{T_{12}}{2\vartheta}w_0z_0\frac{\partial}{\partial t}(V_{2,0}-V^*_{2,0})=0 \tag{3.5}$$

in which, as expected, only the zeroth-order coefficient functions $u_0(t)$, $v_0(t)$, $w_0(t)$ and $z_0(t)$ appear together with the parameters λ, ϑ and T_{ij} $(i, j = 1, 2)$ in a special combination. This condition looks much more complicated than that obtained above (cf with equations (3.1a) and (3.1b)). Moreover both potentials V_1 and V_2 are occurring within a single relation.

Let us now write the external potentials in the form

$$V_1(x,t)=\alpha(x,t)+\mathrm{i}\beta(x,t) \tag{3.6a}$$
$$V_2(x,t)=\gamma(x,t)+\mathrm{i}\delta(x,t) \tag{3.6b}$$

where α, γ and β, δ are real functions. Exploiting the reality of $V_{1,1}$ and $V_{2,1}$ expressed by relation (3.4b) and using the definition (2.7), we obtain the results

$$\beta(x,t)\equiv\beta(t) \qquad \text{and} \qquad \delta(x,t)\equiv\delta(t). \tag{3.7}$$

This condition, which can be checked easily by direct substitution, tells us that the imaginary part of the potential may depend only on the time t and not on the space x variables. Using this last result and inserting the definitions (3.6a) and (3.6b) into the relation (3.5) we obtain the following expression:

$$\left[-\frac{2}{\lambda}T_{21}u_0v_0\beta^2-\frac{2}{\vartheta}T_{12}w_0z_0\delta^2+\frac{1}{\lambda}T_{21}u_0v_0\frac{\mathrm{d}\beta}{\mathrm{d}t}+\frac{1}{\vartheta}T_{12}w_0z_0\frac{\mathrm{d}\delta}{\mathrm{d}t}\right]$$
$$-T_{12}w_0z_0\frac{\partial^2\gamma}{\partial x^2}-T_{21}u_0v_0\frac{\partial^2\alpha}{\partial x^2}=0. \tag{3.8}$$

Because the quantity in the square bracket depends only on time t, integration by x twice yields the following results:

$$T_{12}w_0z_0\gamma+T_{21}u_0v_0\alpha=C_1(t)x^2+C_2(t)x+C_3(t) \tag{3.9}$$

where the coefficients $C_1(t)$, $C_2(t)$ and $C_3(t)$ depend only on time t, and C_2 and C_3 are arbitrary real functions. By re-substituting this latter equation into expression (3.8), we obtain the constraint for the function $C_1(t)$ as follows:

$$C_1(t)=\frac{T_{21}}{\lambda}u_0v_0\left(\frac{1}{2}\frac{\mathrm{d}\beta}{\mathrm{d}t}-\beta^2\right)+\frac{T_{12}}{\vartheta}w_0z_0\left(\frac{1}{2}\frac{\mathrm{d}\delta}{\mathrm{d}t}-\delta^2\right). \tag{3.10}$$

We emphasize that the above result does not mean a restriction for the individual form of the real part of the external potentials V_1 and V_2. As equation (3.9) shows only a weighted sum of the real parts α and γ is constrained by the compatibility conditions (3.5) belonging to the resonance $j = 4$.

Let us now exhibit a possible consequence of the general constraints (3.9) and (3.10) for the potentials by starting from an obvious splitting of the coefficient $C_1(t)$ as follows:

$$C_1(t)=C_1^{(1)}(t)+C_1^{(2)}(t) \tag{3.11}$$

with

$$C_1^{(1)}=\frac{T_{21}u_0v_0}{\lambda}\left(\frac{1}{2}\frac{\mathrm{d}\beta}{\mathrm{d}t}-\beta^2\right) \qquad \text{and} \qquad C_1^{(2)}=\frac{T_{12}w_0z_0}{\vartheta}\left(\frac{1}{2}\frac{\mathrm{d}\delta}{\mathrm{d}t}-\delta^2\right). \tag{3.12a}$$

Then the compatibility condition expressed by equation (3.9) can be satisfied by the following choices:

$$T_{12}w_0z_0\gamma = C_1^{(1)}x^2 + C_2^{(1)}x + C_3^{(1)} + f(x,t) \qquad (3.13a)$$

$$T_{21}u_0v_0\alpha = C_1^{(2)}x^2 + C_2^{(2)}x + C_3^{(2)} - f(x,t) \qquad (3.13b)$$

where $C_i^{(1)}$ and $C_i^{(2)}$ $(i = 2, 3)$ are arbitrary real functions of time t and f is an arbitrary real function of x and t. Using the expressions (3.6a) and (3.6b) we obtain the following possible form for the external potentials:

$$V_1 = \frac{1}{\lambda}\left(\frac{1}{2}\frac{\mathrm{d}\beta(t)}{\mathrm{d}t} - \beta^2(t)\right)x^2 + V_1^{(1)}(t)x + V_1^{(0)}(t) - \frac{\tilde{V}(x,t)}{T_{21}\,(\lambda T_{22} - \vartheta T_{12})} + \mathrm{i}\beta(t) \qquad (3.14a)$$

$$V_2 = \frac{1}{\vartheta}\left(\frac{1}{2}\frac{\mathrm{d}\delta(t)}{\mathrm{d}t} - \delta^2(t)\right)x^2 + V_2^{(1)}(t)x + V_2^{(0)}(t) + \frac{\tilde{V}(x,t)}{T_{12}\,(\vartheta T_{11} - \lambda T_{21})} + \mathrm{i}\delta(t) \qquad (3.14b)$$

where $V_1^{(1)}$, $V_1^{(0)}$, $V_2^{(1)}$ and $V_2^{(0)}$ are arbitrary real functions of time t and $\tilde{V}(x,t)$ represents an arbitrary real potential function which may be used conveniently in design of experiments with BECs. At this point we have to note that these formulae cannot be used for the uncoupled case, since when $T_{12} = T_{21} = 0$, then the compatibility condition (2.22) changes to (2.24ba), (2.24bb) and in this way $\tilde{V}(x,t)$ does not arise.

In summary we conclude that in order for the coupled GP equations (2.3a) and (2.3b) to pass the P test, a special combination of the real parts of the potentials V_1 and V_2 may depend only quadratically and/or linearly on the spatial coordinate x. A stringent relationship can be established between the coefficient of the quadratic terms and the imaginary parts, which, in turn, may depend only on time t. An additional real potential $\tilde{V}$ of general form may be introduced which explicitly exhibits coupling between the external potentials V_1 and V_2.

4. Discussion of the results

In this section we discuss the results from various points of view and make comparison with related results obtained by others.

4.1. Presence of source terms

In the course of the theoretical study of two-component BECs with attractive interaction, it has been found [22] that the decay and growth of number of atoms is best accounted for by introducing an imaginary contact interaction term in the GP equations. We now see that our analysis enables the existence of such source terms, by appropriately chosen $\beta(t)$ and $\delta(t)$ (see equations (3.14a), (3.14b)). This result holds also in the case of one-component BECs as found by Clarkson [30].

4.2. Uncoupled case

Next, we investigate the case $T_{12} = T_{21} \equiv 0$, when the system of the GP equations (2.3a) and (2.3b) is decoupled. As an example we derive the resonances. Our general equations should reduce twice to earlier results obtained by Clarkson [30] for the one-component GP equation. Starting from the general expression (2.14) and applying the useful formula (2.12) one obtains

$$\det\big(Q(j)\big) = \lambda^2\vartheta^2(j+1)\,j^2(j-3)^2(j-4)\left(j^2 - 3j + 4 - 2\frac{T_{11}T_{22}}{\Delta}\frac{2\lambda\vartheta + 2\lambda\vartheta}{\lambda\vartheta}\right)$$
$$= \lambda^2\vartheta^2(j+1)\,j^2(j-3)^2(j-4)(j^2 - 3j - 4)$$

$$= \lambda^2 \vartheta^2 (j+1)^2 j^2 (j-3)^2 (j-4)^2. \tag{4.1}$$

The resonances $(-1, 0, 3, 4)$ are those found by Clarkson [30] and all have a multiplicity of two as a result of the double number of the (uncoupled) GP equations.

4.3. Sign of the potential

One of the experimental situations where the coupled GP equations (2.3*a*) and (2.3*b*) serve as a theoretical basis is the creation of two-component BECs [16]. In such experiments alkali atoms are confined by symmetrically arranged harmonic trap potentials. One of the possibilities of our results is to choose in equations (3.14*a*) and (3.14*b*) all functions $V_1^{(1)}$, $V_1^{(0)}$, $V_2^{(1)}$, $V_2^{(0)}$, β and δ equal to zero and let the potential $\tilde{V}(x,t)$ operate as a field confining the alkali gas particles. It is then required that in equations (3.14*a*) and (3.14*b*) the terms in which our confining potential $\tilde{V}$ occurs have the same sign. The condition that those two terms with $\tilde{V}$ have the same sign is in general

$$T_{12}T_{21}\,(\lambda T_{22} - \vartheta T_{12})\,(\vartheta T_{11} - \lambda T_{21}) < 0 \tag{4.2a}$$

which can be expressed also in terms of the scattering lengths as

$$a_{12}a_{21}\,(\lambda a_{22} - \vartheta a_{12})\,(\vartheta a_{11} - \lambda a_{21}) < 0. \tag{4.2b}$$

Because, physically, $a_{12} = a_{21}$, the above condition for the equality of signs of the $\tilde{V}$ terms in equations (3.14*a*), (3.14*b*) is fulfilled for the usual experimental case with $\lambda = \vartheta$ if

$$a_{11} < a_{12} < a_{22} \qquad \text{or} \qquad a_{22} < a_{12} < a_{11}. \tag{4.3}$$

If the scattering lengths $a_{12} = a_{21}$ are greater or less than both a_{11} and a_{22} then the sign of the terms containing the arbitrary potential $\tilde{V}$ is different, which corresponds to untrapping one of the BEC components.

4.4. Fulfillment of equation (2.18)

The best studied example of the two-component BECs involves Rb atoms in two different hyperfine states. It has been found experimentally [7, 8, 16], and numerically [18] that a stable configuration of soliton-like vortex in the two-component condensate is achieved in the case where the scattering lengths are in the proportion:

$$a_{11} : a_{12} : a_{22} = 1.03 : 1 : 0.97 \qquad \text{with} \quad a_{12} \equiv a_{21}. \tag{4.4}$$

Let us now check whether these ratios obey our general condition (2.18) with integer m. Since $\lambda/\vartheta = 1$ expression (2.18) can be written as follows:

$$\frac{2(a_{11}/a_{21})(a_{22}/a_{12}) - (a_{11}/a_{21}) - (a_{22}/a_{12})}{(a_{11}/a_{21})(a_{22}/a_{12}) - 1} = \frac{1}{16}\left[(2m+1)^2 + 7\right]. \tag{4.5}$$

The insertion of the above ratios gives

$$\frac{2 \cdot 1.03 \cdot 0.97 - 1.03 - 0.97}{1.03 \cdot 0.97 - 1} = \frac{-0.0018}{-0.0009} = 2 \equiv \frac{1}{16}\left[(2m+1)^2 + 7\right] \tag{4.6}$$

which yields

$$m = 2. \tag{4.7}$$

This result means that the experimental ratios (4.4) correspond to just one of the possible solutions of the GP equations characterized by a $m = 2$ potential family.

Proceeding further, one can determine the resonances belonging to the experimentally found ratios (4.4) to be (cf with equations (2.16) and (2.17))

$$j_1 = 2 + m = 4 \qquad j_2 = 1 - m = -1. \tag{4.8}$$

This result means that no further work is needed, the underlying potential falls into the category defined by the compatibility condition for $j = 4$; a possible representation of such potentials is given by (3.14a) and (3.14b). Indeed, the quadratic trap potential used in the experiments suits well the general form of potentials obtained from the analysis of the resonance at $j = 4$.

5. Summary

In this paper the first step towards verification of integrability of the coupled GP equations by means of the P analysis has been carried out. It has been shown that the GP equations pass the P test provided a special relation among the system parameters (masses and interaction strengths) is satisfied (cf with (2.18)). One of the recent experiments has been taken as an example. In this experiment [7, 8, 16] and a subsequent numerical study [18], the vortex stability of a two-component BEC has been investigated. It is found that the system parameters at which stability occurs are just in the proportion which fits our relation (2.18) with $m = 2$, a condition necessary for the GP equation to pass the P test.

As the GP equations play a great role in describing BECs, particular attention has been paid to establishing the admissible forms of the confining trap potentials of experimental interest. It has been found that, in addition to the prescribed form resulting from the P analysis of a single GP equation, there is a possibility of introducing an extra potential term of arbitrary shape into the external potentials (cf with equations (3.14a) and (3.14b)). Also, some discussion of the results has been added, which includes the comparison of the earlier results obtained for the one-component GP equation, the role of the source (imaginary) terms $\beta(t)$ and $\delta(t)$ in the potentials, and the sign of the additional potential terms.

Finally, we add a remark to the fulfillment of equation (2.18) with integer m. In the light of experimental errors the above agreement $m = 2$ may seem to be accidental. We note however that soliton-type structures (e.g. vortices in 3D) possess an outstanding stability sometimes called 'robustness', which enables these particle-like formations to survive for a long time or even to arise in circumstances that do not fit the exact constraint of mathematics. Therefore equation (2.18) may prove also useful in exploring other regions of parameters where such stable structures are to be observed in binary condensates.

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